Decoupling of OF equations for solution of 4D images with Gauss-Seidel equations

Having applied Euler-Lagrange equations to minimise the error function, one obtains the matrix equation 1.

$$\begin{bmatrix} (E_x^2+\alpha^2) & E_xE_y & E_xE_z \\ E_xE_y & (E_y^2+\alpha^2) & E_yE_z \\ E_xE_z & E_yE_z & (E_z^2+\alpha^2) \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \alpha^2\bar{u}_{i,j,k} - E_xE_t \\ \alpha^2\bar{v}_{i,j,k} - E_yE_t \\ \alpha^2\bar{v}_{i,j,k} - E_zE_t \end{bmatrix} \tag{1}$$

Equation 1 can be rearranged to form the following equations

$$(\alpha^2 + E_x^2)u + E_xE_yv + E_xE_zw = \alpha^2\bar{u} - E_xE_t \tag{2}$$

$$E_xE_yu + (\alpha^2 + E_y^2)v + E_yE_zw = \alpha^2\bar{v} - E_yE_t \tag{3}$$

$$E_xE_zu + E_yE_zv + (\alpha^2 + E_z^2)w = \alpha^2\bar{w} - E_zE_t \tag{4}$$

multiply Equation 2 by E_y and Equation 3 by E_x and subtract to eliminate w:

$$\begin{aligned} E_y(\alpha^2 + E_x^2)u + E_xE_y^2v + E_xE_yE_zw &= \alpha^2E_y\bar{u} - E_xE_yE_t \\ E_x^2E_yu + E_x(\alpha^2 + E_y^2)v + E_xE_yE_zw &= \alpha^2E_x\bar{v} - E_xE_yE_t \\ E_yu - E_xv &= E_y\bar{u} - E_x\bar{v} \end{aligned}$$

Eliminate v from Equations 2 and 4

$$\begin{aligned} E_z(\alpha^2 + E_x^2)u + E_xE_yE_zv + E_xE_z^2w &= \alpha^2E_z\bar{u} - E_xE_zE_t \\ E_x^2E_zu + E_xE_yE_zv + E_x(\alpha^2 + E_z^2)w &= \alpha^2E_x\bar{w} - E_xE_zE_t \\ E_zu - E_xw &= E_z\bar{u} - E_x\bar{w} \\ E_zu &= E_z\bar{u} + E_x(w - \bar{w}) \end{aligned}$$

Eliminate u from Equations 3 and 4

$$\begin{aligned} E_xE_yE_zu + E_z(\alpha^2 + E_y^2)v + E_yE_z^2w &= \alpha^2E_z\bar{v} - E_yE_zE_t \\ E_xE_yE_zu + E_y^2E_zv + E_y(\alpha^2 + E_z^2)w &= \alpha^2E_y\bar{w} - E_yE_zE_t \\ E_zv - E_yw &= E_z\bar{v} - E_y\bar{w} \\ E_zv &= E_z\bar{v} + E_y(w - \bar{w}) \end{aligned}$$

Substitute into Equation 4:

$$\begin{aligned} E_xE_zu + E_yE_zv + (\alpha^2 + E_z^2)w &= \alpha^2\bar{w} - E_zE_t \\ E_x(E_x(w - \bar{w}) + E_z\bar{u}) + E_y(E_y(w - \bar{w}) + E_z\bar{v}) + (\alpha^2 + E_z^2)w &= \alpha^2\bar{w} - E_zE_t \\ E_x^2(w - \bar{w}) + E_xE_z\bar{u} + E_y^2(w - \bar{w}) + E_yE_z\bar{v} + (\alpha^2 + E_z^2)w &= \alpha^2\bar{w} - E_zE_t \\ w(E_x^2 + E_y^2 + E_z^2 + \alpha^2) - \bar{w}(E_x^2 + E_y^2 + \alpha^2) &= -E_xE_z\bar{u} - E_yE_z\bar{v} - E_zE_t \\ (w - \bar{w})(E_x^2 + E_y^2 + E_z^2 + \alpha^2) &= -E_xE_z\bar{u} - E_yE_z\bar{v} - E_z^2\bar{w} - E_zE_t \end{aligned}$$

Rearranging gives the Equation 5, expressing velocity component w in terms of value of α^2, partial derivatives of image intensities and average velocities.

$$w = \bar{w} - \frac{E_xE_z\bar{u} + E_yE_z\bar{v} + E_z^2\bar{w} + E_zE_t}{(E_x^2 + E_y^2 + E_z^2 + \alpha^2)} \tag{5}$$

Similarly, put u in terms of $\bar{u}, \bar{v}, \bar{w}$

$$\begin{aligned}
(\alpha^2 + E_x^2)u + E_y E_x v + E_z E_x w &= \alpha^2 \bar{u} - E_x E_t \\
(\alpha^2 + E_x^2)u + E_y(E_x \bar{v} + E_y(u - \bar{u})) + E_z(E_x \bar{w} + E_z(u - \bar{u})) &= \alpha^2 \bar{u} - E_x E_t \\
u(\alpha^2 + E_x^2 + E_y^2 + E_z^2) - \bar{u}(\alpha^2 + E_y^2 + E_z^2) &= -(E_x E_t + E_x E_y \bar{v} + E_x E_z \bar{w}) \\
(u - \bar{u})(\alpha^2 + E_x^2 + E_y^2 + E_z^2) &= -(E_x^2 \bar{u} + E_x E_y \bar{v} + E_x E_z \bar{w} + E_x E_t)
\end{aligned}$$

Rearranging gives Equation 6, expressing velocity component u in terms of value of α^2, partial derivatives of image intensities and average velocities.

$$u = \bar{u} - \frac{E_x^2 \bar{u} + E_x E_y \bar{v} + E_x E_z \bar{w} + E_x E_t}{(E_x^2 + E_y^2 + E_z^2 + \alpha^2)} \tag{6}$$

Equation 7 can be found similarly, expressing velocity component v in terms of value of α^2, partial derivatives of image intensities and average velocities.

$$v = \bar{v} - \frac{E_y E_x \bar{u} + E_y^2 \bar{v} + E_y E_z \bar{w} + E_y E_t}{(E_x^2 + E_y^2 + E_z^2 + \alpha^2)} \tag{7}$$

Gauss Seidel equation can be derived by simply updating values of u, v, w with old estimates of $\bar{u}, \bar{v}, \bar{w}$, producing Equations 8, 9 and 10.

$$u^{n+1} = \bar{u}^n - \frac{E_x^2 \bar{u}^n + E_x E_y \bar{v}^n + E_x E_z \bar{w}^n + E_x E_t}{(E_x^2 + E_y^2 + E_z^2 + \alpha^2)} \tag{8}$$

Similarly,

$$v^{n+1} = \bar{v}^n - \frac{E_y E_x \bar{u}^n + E_y^2 \bar{v}^n + E_y E_z \bar{w}^n + E_y E_t}{(E_x^2 + E_y^2 + E_z^2 + \alpha^2)} \tag{9}$$

$$w^{n+1} = \bar{w}^n - \frac{E_x E_z \bar{u}^n + E_y E_z \bar{v}^n + E_z^2 \bar{w}^n + E_z E_t}{(E_x^2 + E_y^2 + E_z^2 + \alpha^2)} \tag{10}$$